

# EXACT AND EFFICIENT SIMULATION OF TAIL PROBABILITIES OF HEAVY-TAILED INFINITE SERIES

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**ABSTRACT.** We develop an efficient simulation algorithm for computing the tail probabilities of the infinite series  $S = \sum_{n \geq 1} a_n X_n$  when random variables  $X_n$  are heavy-tailed. As  $S$  is the sum of infinitely many random variables, any simulation algorithm that stops after simulating only fixed, finitely many random variables is likely to introduce a bias. We overcome this challenge by rewriting the tail probability of interest as a sum of a random number of telescoping terms, and subsequently developing conditional Monte Carlo based low variance simulation estimators for each telescoping term. The resulting algorithm is proved to result in estimators that a) have no bias, and b) require only a fixed, finite number of replications irrespective of how rare the tail probability of interest is. Thus, by combining a traditional variance reduction technique such as conditional Monte Carlo with more recent use of auxiliary randomization to remove bias in a multi-level type representation, we develop an efficient and unbiased simulation algorithm for tail probabilities of  $S$ . These have many applications including in analysis of financial time-series and stochastic recurrence equations arising in models in actuarial risk and population biology.

## 1. INTRODUCTION

Given a sequence of regularly varying random variables  $(X_n : n \geq 1)$  and discount factors  $(a_n : n \geq 1)$ , the objective of this paper is to design an algorithm that computes tail probabilities of linear models of form,

$$S = \sum_{n \geq 1} a_n X_n.$$

In addition to arising naturally in the study of linear processes and stochastic recurrence equations, such infinite series are also used in risk analysis to model instances where, for example, the surplus of an insurance firm is invested in a risky asset. See [20, 35, 18, 30, 29] and references therein for a review of stochastic models where the infinite series  $S$  is a central object of interest.

Since exact computation of  $P(S > b)$ , for a given positive real number  $b$ , is generally not possible, it is common to resort to Monte Carlo simulations. However, as the object of interest involves infinitely many random variables, any simulation algorithm that stops after generating only finitely many random variables is likely to introduce a bias. In addition, as the parameter  $b$  increases, the event of interest,  $\{S > b\}$ , becomes more rare, thus making the problem harder to estimate within a limited computational budget. The objective of this paper is to design a Monte Carlo algorithm that resolves these difficulties. Precisely, we design a family of simulation estimators  $(Z(b) : b > 0)$  for estimating probabilities  $P(S > b)$  such that,

- 1) the estimators have no bias,
- 2) the variance of the family  $Z(b)$  is uniformly bounded.

These two properties, in turn, help in guaranteeing that the output of the Monte Carlo procedure is within a pre-specified relative precision after only expending an expected computational effort that is uniformly bounded in  $b$ . In other words, the expected computational effort remains bounded irrespective of the rarity of the event.

While the study on bias minimization in Monte Carlo simulations received a huge boost with the introduction of multi-level simulation (see [16]), the prospect of eliminating bias altogether has become possible with the debiasing techniques employed in [26, 31] and [28]). As we quickly illustrate in Section 3, use of a suitably chosen auxiliary random variable that determines when the algorithm terminates is at the heart of these new class of algorithms that eliminate bias. For our simulation problem, this technique enables us to work on modified ‘local’ problems only involving random variables  $a_i, X_i$ , for  $i$  not exceeding a random level  $N$ . Then the probability law of this random level  $N$  is chosen carefully in order to combine estimators for these local problems without bias.

Though the use of a suitably chosen auxiliary random variable may eliminate bias, it is not sufficient to deal with the fact that the probabilities  $P(S > b)$  are small for large values of  $b$ . Consequently, a ‘naive’ simulation algorithm will require as many as  $O(1/P(S > b))$  repeated simulation runs to achieve a desired relative precision (see [23]). Since this is computationally expensive, we propose new Monte Carlo estimators that are effective for simulating rare events of our interest. More precisely, we devise a family of conditional Monte Carlo estimators  $(Z_{\text{loc}}(n, b) : n \geq 1)$ , for a given threshold  $b$ , to solve the family of local problems indexed by  $n$ . Here, recall that the  $n$ -th local problem is such that it involves only random variables  $(a_i, X_i : i = 1, \dots, n)$ , and hence can be solved in finite time. Then, assuming that the random variables  $X_i$  are regularly varying, we show that the variance of the local estimators  $Z_{\text{loc}}(n, b)$  are sufficiently low, uniformly in  $n$  and  $b$ .

By carefully choosing the law of the auxiliary random variable  $N$ , we combine these local estimators  $Z_{\text{loc}}(n, b)$  to develop an unbiased estimator for  $P(S > b)$  with bounded coefficient of variation (relative error). As we show, this ensures that the computational complexity does not scale with  $b$ , even if the probability  $P(S > b)$  becomes rare. In addition, we verify that the expected termination time of the simulation algorithm is finite, thereby guaranteeing that the estimation can be performed with a computational effort that is uniformly bounded in  $b$ .

As estimation of tail probabilities of heavy-tailed sums has been known to be more challenging than their light-tailed counterparts (see [1, 5]), simulation algorithms using a variety of techniques (such as conditional Monte Carlo (see [3]), importance sampling (see [1, 22, 21, 14, 6, 9, 10, 28]), splitting [11], Markov chain Monte Carlo [19], and cross-entropy method [12]) have been developed after intense research over previous decade. In particular, [7] develops an importance sampling algorithm for simulation of tail probabilities of the stochastic recurrence equations of the form

$$X_{n+1} = A_{n+1}X_n + B_{n+1}, \quad X_0 = 0,$$

for large values of  $n$ . As noted by them, this has applications in a variety of settings ranging from financial time series, actuarial risk and population biology (see [24, 17, 32, 4, 25, 34] and references therein). As  $n \rightarrow \infty$ , the distribution of  $X_n$  corresponds to the stationary distribution of the Markov chain modeled by  $X_n$ , and can be represented as an infinite series (see [20]). While importance sampling remains the most studied technique for simulation of such tail probabilities, conditional Monte Carlo estimators devised by Asmussen and Kroese [3] have been shown to offer superior numerical accuracy (see Section 3.5 in [27]). Leveraging this, we design intuitive, easy-to-use Asmussen-Kroese type conditional Monte Carlo estimators to solve

the local problems mentioned earlier<sup>1</sup>. As the sampling techniques involved for simulating rare events in light-tailed sums are drastically different (see, for example, [33, 23, 8]), we note that a similar study for estimation of tail probabilities of light-tailed infinite sums as an interesting future research direction.

The paper is organized as follows: After describing the problem of interest precisely in Section 2, we develop the simulation methodology in Section 3. A detailed variance analysis that characterizes the computational complexity of the family of local estimators and the overall estimators introduced in Section 3 is presented in Section 4. A numerical example that reaffirms the theoretical efficiency results of the paper is presented in Section 5. Technical proofs that are not central to the variance analysis in Section 4 are presented in the appendix.

## 2. NOTATIONS AND PROBLEM STATEMENT

To precisely introduce the problem, let  $X$  be a zero mean random variable satisfying the following condition:

**Assumption 1.** *The distribution function of  $X$ , denoted by  $F(\cdot)$ , is such that the tail probabilities  $\bar{F}(x) := 1 - F(x) = x^{-\alpha}L(x)$  for some slowly varying function  $L(\cdot)$  and  $\alpha > 2$ .*

Here, the slowly varying function  $L(\cdot)$  stands for any function that satisfies  $L(tx)/L(x) \rightarrow 1$ , for every  $t > 0$ , as  $x \rightarrow \infty$ . When  $L(\cdot) = c$  for some positive constant  $c$ , we obtain the special case of Pareto (or) power-law distributions. Other common examples of slowly varying functions include logarithmically decaying/growing functions such as  $\log x, \log \log x, 1/\log x$ , etc. The following property, commonly referred as Potter's bounds, confirms that regularly varying tail probabilities  $\bar{F}(\cdot)$  are essentially polynomially decaying: there exists a  $t_\delta > 0$  such that for all  $t$  and  $v$  satisfying  $t \geq t_\delta$  and  $vt \geq t_\delta$ ,

$$(1) \quad (1 - \delta) \min\{v^{-\alpha+\delta}, v^{-\alpha-\delta}\} \leq \frac{\bar{F}(vx)}{\bar{F}(x)} \leq (1 + \delta) \max\{v^{-\alpha+\delta}, v^{-\alpha-\delta}\}.$$

See, for example, Chapter VIII of [15] or Chapter 1 of [13], for a proof of (1), and other important properties of regularly varying distributions.

Let  $(X_n : n \geq 1)$  be a sequence of i.i.d. copies of  $X$ . Our aim is to efficiently estimate the tail probabilities of

$$S := \sum_n a_n X_n,$$

where  $(a_n : n \geq 1)$  satisfies the following condition:

**Assumption 2.** *The sequence  $(a_n : n \geq 1)$  is such that  $a_n$  lies in the interval  $(0, 1)$  for every  $n$  and  $\sum_n na_n < \infty$ .*

The random variable  $S$  is proper because  $\sum_n a_n^2 < \infty$  (follows from Kolmogorov's three-series theorem). The assumptions that  $X$  has zero mean and  $a_n < 1$  have been made just for the ease of exposition. If  $X$  has non-zero mean or if  $a_n > 1$  for any  $n$ , then the corresponding problem of estimating  $P\{S > b\}$  can be translated to a problem instance satisfying Assumptions 1 and

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<sup>1</sup>The proposed estimators and their variance analysis comprise Chapter 5 in the PhD Dissertation [27] of one of the authors

2 by letting  $\tilde{a}_n := a_n / \sup_n a_n$  and by instead simulating the right hand side of the equation below:

$$P \left\{ \sum_n a_n X_n > b \right\} = P \left\{ \sum_n \tilde{a}_n (X_n - EX) > \frac{b - (\sum_n a_n) EX}{\sup_n a_n} \right\}.$$

Here note that  $\sup_n a_n$  exists because we require  $\sum_n a_n < \infty$ .

### 3. SIMULATION METHODOLOGY

Given  $b > 0$ , we aim to estimate  $P\{S > b\}$  via simulation. If  $S$  is a sum of, say, for example,  $k$  i.i.d. random variables  $X_1, \dots, X_k$ , then one can simply generate an i.i.d. realization of  $X_1, \dots, X_k$  and check whether their sum is larger than  $b$  or not. However, the countably infinite number of random variables involved in the definition of  $S$  makes the task of obtaining a sample of  $S$  via its increments, at least at a preliminary look, appear computationally infeasible. To overcome this difficulty, we introduce an auxiliary random variable  $N$  and re-express the probability  $P\{S > b\}$  below in (2) in a form that gives computational tractability: Let

$$S_0 := 0 \text{ and } S_n := \sum_{i=1}^n a_i X_i \text{ for } n \geq 1.$$

Further, let  $p_n := P\{N = n\}$  be positive for every  $n \geq 1$ . Then,

$$\begin{aligned} P\{S > b\} &= \lim_n P\{S_n > b\} \\ &= \sum_{n \geq 1} p_n \frac{P\{S_n > b\} - P\{S_{n-1} > b\}}{p_n} \\ (2) \quad &= E \left[ \frac{P\{S_N > b \mid N\} - P\{S_{N-1} > b \mid N\}}{p_N} \right] \end{aligned}$$

In Section 3.1, we aim to develop unbiased estimators  $(Z_{\text{loc}}(n, b) : n \geq 1, b > 0)$  satisfying the following desirable properties:

- (1) The expectation of  $Z_{\text{loc}}(n, b)$  is  $P\{S_n > b\} - P\{S_{n-1} > b\}$  for every  $n$  and  $b$ .
- (2) The computational effort required to generate a realization of  $Z_{\text{loc}}(n, b)$  is bounded from above by  $Cn$ , for some constant  $C > 0$ , uniformly for all  $b$ .
- (3) The estimators  $Z_{\text{loc}}(n, b)$  have low variance, uniformly in  $n$  and  $b$ .

Now, in a simulation run, if the realized value of  $N$  is  $n$ , we generate an independent realization of estimator  $Z_{\text{loc}}(n, b)$  and use

$$Z(b) := \frac{Z_{\text{loc}}(N, b)}{p_N}$$

as an estimator for  $P\{S > b\}$ . The fact that  $Z(b)$  yields estimates of  $P\{S > b\}$  without any bias follows from (2). Thus by introducing an auxiliary random variable  $N$ , in every simulation run, we are faced with the task of generating only finitely many random variables, as opposed to the naive approach which requires generation of countably infinite random variables. The random variables  $Z_{\text{loc}}(n, b)$  which are instrumental in estimating the tail probabilities of  $S$  will be referred hereafter as ‘local’ estimators.

**3.1. Local estimators.** As mentioned before, in this section, we present estimators for quantities

$$(P\{S_n > b\} - P\{S_{n-1} > b\} : n \geq 1)$$

that have low variance, uniformly in  $n$ , as  $b \rightarrow \infty$ . These form building blocks to serve our initial aim of estimating the tail probabilities of  $S$ . It is well-known that the sum of heavy-tailed random variables attain a large value typically because one of the increments (and hence the maximum of the increments) attain a large value. Therefore, we focus our attention on identifying the maximum of the increments

$$M_n := \max\{a_i X_i : 1 \leq i \leq n\}$$

in a manner that is reflective of the way in which the rare event under consideration happens. For this, we partition the sample space based on which of the  $n$  increments  $\{a_1 X_1, \dots, a_n X_n\}$  is the maximum. Let  $\text{Max}_n$  denote the index of the increment  $a_i X_i$  that equals the maximum  $M_n$ . In case of many increments having the same value as the maximum, we take the largest (index) of them to be  $\text{Max}_n$ . That is,

$$\text{Max}_n := \max\{\text{argmax}\{a_i X_i : 1 \leq i \leq n\}\}.$$

See that the quantity  $P\{S_n > b\} - P\{S_{n-1} > b\}$  can be alternatively expressed as

$$(3) \quad P\{S_n > b\} - P\{S_{n-1} > b\} = p_1(n, b) + p_2(n, b)$$

where

$$\begin{aligned} p_1(n, b) &= P\{S_n > b, \text{Max}_n = n\} - P\{S_{n-1} > b, \text{Max}_n = n\} \text{ and} \\ p_2(n, b) &= P\{S_n > b, \text{Max}_n \neq n\} - P\{S_{n-1} > b, \text{Max}_n \neq n\}. \end{aligned}$$

We develop alternative representations for quantities  $p_1(n, b)$  and  $p_2(n, b)$  and use them to separately estimate  $p_1(n, b)$  and  $p_2(n, b)$  in the following sections.

**3.1.1. Estimator for  $p_1(n, b)$ .** Observe that  $P\{S_{n-1} > b, S_n \leq b, \text{Max}_n = n\} = 0$  because whenever  $S_n \leq b$  and  $S_{n-1} > b$ , it is necessary that  $X_n$  be negative, and in which case  $S_n$  also needs to be negative (since  $M_n = a_n X_n$ ). Therefore,

$$\begin{aligned} p_1(n, b) &= P\{S_n > b, S_{n-1} \leq b, \text{Max}_n = n\} - P\{S_{n-1} > b, S_n \leq b, \text{Max}_n = n\} \\ &= P\{S_n > b, S_{n-1} \leq b, \text{Max}_n = n\}. \end{aligned}$$

Further,

$$\begin{aligned} P\{S_n > b, \text{Max}_n = n \mid X_1, \dots, X_{n-1}\} &= P\{a_n X_n > b - S_{n-1}, a_n X_n > M_{n-1} \mid X_1, \dots, X_{n-1}\} \\ &= \bar{F}\left(\frac{1}{a_n} ((b - S_{n-1}) \vee M_{n-1})\right). \end{aligned}$$

Therefore, it is immediate that

$$E\left[\bar{F}\left(\frac{1}{a_n} ((b - S_{n-1}) \vee M_{n-1})\right) \mathbb{I}(S_{n-1} \leq b)\right] = P\{S_n > b, S_{n-1} \leq b, \text{Max}_n = n\}.$$

If we let

$$Z_1(n, b) := \bar{F}\left(\frac{1}{a_n} ((b - S_{n-1}) \vee M_{n-1})\right) \mathbb{I}(S_{n-1} \leq b),$$

then it follows from the above discussion that  $E[Z_1(n, b)]$  equals  $p_1(n, b)$ . We note this observation below as Lemma 1.

**Lemma 1.** For every  $n > 1$  and  $b > 0$ ,  $E[Z_1(n, b)] = p_1(n, b)$ .

In a simulation run, one can generate samples of  $X_1, \dots, X_{n-1}$  simply from the distribution  $F(\cdot)$  and plug it in the expression of  $Z_1(n, b)$  to arrive at an unbiased estimator for  $p_1(n, b)$ . Since  $Z_1(n, b)$  is just the probability that the event of interest  $\{S_n > b, S_{n-1} \leq b, \text{Max}_n = n\}$  happens conditional on the observed values of  $X_1, \dots, X_{n-1}$ ,  $Z_1(n, b)$  is said to belong to a family of estimators called conditional Monte Carlo estimators (see, for example, [2]). Estimators of the form  $Z_1(n, b)$ , also referred to as Asmussen-Kroese estimators, are shown to be extremely effective in the simulation of tail probabilities of sums of fixed number of heavy-tailed random variables in [3].

**3.1.2. Estimator for  $p_2(n, b)$ .** Similar to  $p_1(n, b)$ , one can develop conditional Monte Carlo estimators for the simulation of  $p_2(n, b)$  as well. To accomplish this, we need more notation: For any  $j \leq n$ , let

$$S_n^{(-j)} := \sum_{i=1, i \neq j}^n a_i X_i \text{ and } M_n^{(-j)} := \max_{i \leq n, i \neq j} a_i X_i.$$

Further, for any  $n > 1$ , let  $(q(j, n) : 0 < j < n)$  be a probability mass function that assigns positive probability to every integer in  $\{1, \dots, n-1\}$ . Let  $J_n$  be an auxiliary random variable which takes values in  $\{1, \dots, n-1\}$  such that  $P\{J_n = j\} = q(j, n)$ . Aided with this notation, define the estimator for  $p_2(n, b)$  as

$$Z_2(n, b) := \frac{Z_{2,1}(n, b) - Z_{2,2}(n, b)}{q(J_n, n)},$$

where

$$\begin{aligned} Z_{2,1}(n, b) &:= \bar{F} \left( \frac{1}{a_{J_n}} \left( (b - S_n^{(-J_n)}) \vee M_n^{(-J_n)} \right) \right) \text{ and} \\ Z_{2,2}(n, b) &:= \bar{F} \left( \frac{1}{a_{J_n}} \left( (b - S_{n-1}^{(-J_n)}) \vee M_n^{(-J_n)} \right) \right). \end{aligned}$$

Lemma 2 below verifies that  $Z_2(n, b)$  is an unbiased estimator for  $p_2(n, b)$ .

**Lemma 2.** *For every  $n > 1$  and  $b > 0$ ,  $E[Z_2(n, b)] = p_2(n, b)$ .*

*Proof.* For any  $n$  and  $j < n$ , observe that

$$\begin{aligned} P \left\{ S_n > b, \text{Max}_n = j \mid S_n^{(-j)}, M_n^{(-j)} \right\} &= P \left\{ a_j X_j > b - S_n^{(-j)}, a_j X_j > M_n^{(-j)} \mid S_n^{(-j)}, M_n^{(-j)} \right\} \\ (4) \qquad \qquad \qquad &= \bar{F} \left( \frac{1}{a_j} \left( (b - S_n^{(-j)}) \vee M_n^{(-j)} \right) \right), \end{aligned}$$

and similarly,

$$(5) \qquad P \left\{ S_{n-1} > b, \text{Max}_n = j \mid S_{n-1}^{(-j)}, M_n^{(-j)} \right\} = \bar{F} \left( \frac{1}{a_j} \left( (b - S_{n-1}^{(-j)}) \vee M_n^{(-j)} \right) \right).$$

Recall that  $J_n$  takes values only in  $\{1, \dots, n-1\}$ . Therefore, it follows from the definition of  $Z_{2,1}(n, b)$  and  $Z_{2,2}(n, b)$  that

$$\begin{aligned} Z_{2,1}(n, b) &= P \left\{ S_n > b, \text{Max}_n = J_n \mid S_n^{(-J_n)}, M_n^{(-J_n)}, J_n \right\} \text{ and} \\ Z_{2,2}(n, b) &= P \left\{ S_{n-1} > b, \text{Max}_n = J_n \mid S_n^{(-J_n)}, M_n^{(-J_n)}, J_n \right\}. \end{aligned}$$

Then it is immediate that

$$\begin{aligned} E[Z_{2,1}(n, b) \mid J_n] &= P\{S_n > b, \text{Max}_n = J_n \mid J_n\} \text{ and} \\ E[Z_{2,2}(n, b) \mid J_n] &= P\{S_{n-1} > b, \text{Max}_n = J_n \mid J_n\}. \end{aligned}$$

Since  $P\{J_n = j\} = q(j, n)$ , it follows that

$$\begin{aligned} (6) \quad E\left[\frac{Z_{2,1}(n, b)}{q(J_n, n)}\right] &= \sum_{j=1}^{n-1} P\{J_n = j\} E\left[\frac{Z_{2,1}(n, b)}{q(J_n, n)} \mid J_n = j\right] \\ &= \sum_{j=1}^{n-1} q(j, n) \frac{E[Z_{2,1}(n, b) \mid J_n = j]}{q(j, n)} \\ &= \sum_{j=1}^{n-1} P\{S_n > b, \text{Max}_n = j\} \\ (7) \quad &= P\{S_n > b, \text{Max}_n \neq n\}. \end{aligned}$$

Similarly one can derive that

$$(8) \quad E\left[\frac{Z_{2,2}(n, b)}{q(J_n, n)}\right] = P\{S_{n-1} > b, \text{Max}_n \neq n\}$$

Since  $Z_2(n, b) = (Z_{2,1}(n, b) - Z_{2,2}(n, b))/q(J_n, n)$ , it is immediate from (7) and (8) that  $E[Z_2(n, b)] = p_2(n, b)$ .  $\square$

To summarize the simulation procedure, we present Algorithm 1 here, which returns a realization of

$$Z_{\text{loc}}(n, b) := Z_1(n, b) + Z_2(n, b)$$

for given values of  $n$  and  $b$ . It follows from Lemmas 1 and 2 that  $Z_{\text{loc}}(n, b)$  is indeed an unbiased estimator for the quantity  $P\{S_n > b\} - P\{S_{n-1} > b\}$ .

**3.2. Simulation of  $P\{S > b\}$ .** We use an auxiliary random variable  $N$  to estimate the tail probabilities of the infinite series  $S = \sum_n a_n X_n$ . Recall that `LOCALSIMULATION`( $n, b$ ) is a simulation procedure introduced in Algorithm 1 in Section 3.1, which for given values of  $n$  and  $b$ , returns realizations of random variable  $Z_{\text{loc}}(n, b)$  that has  $P\{S_n > b\} - P\{S_{n-1} > b\}$  as its expectation. Given  $b > 0$ , we present below Algorithm 2 that makes a call to `LOCALSIMULATION` procedure of Algorithm 1 and returns

$$Z(b) := \frac{Z_{\text{loc}}(N, b)}{p_N}$$

which is the estimator we propose for computing the probability  $P\{S > b\}$ .

**Theorem 3.** *The estimators  $(Z(b) : b > 0)$  are unbiased: that is, for every  $b > 0$ ,*

$$E[Z(b)] = P\{S > b\}.$$

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**Algorithm 1** Given  $n$  and  $b$ , the aim is to efficiently simulate  $P\{S_n > b\} - P\{S_{n-1} > b\}$

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**procedure** LOCALSIMULATION( $n, b$ )

Let  $Z_1(n, b) = \text{ESTIMATOR1}(n, b)$  and  $Z_2(n, b) = \text{ESTIMATOR2}(n, b)$

Return  $Z_{\text{loc}}(n, b) = Z_1(n, b) + Z_2(n, b)$

**procedure** ESTIMATOR1( $n, b$ )

Initialize  $Z_1(n, b) = 0$

Simulate a realization of  $(X_i : 1 \leq i \leq n-1)$  independently from the distribution  $F(\cdot)$

Let  $S_{n-1} = \sum_{i=1}^{n-1} a_i X_i$  and  $M_{n-1} = \max\{a_i X_i : 1 \leq i \leq n-1\}$

**if**  $S_{n-1} \leq b$  **then**

Let

$$Z_1(n, b) = \bar{F} \left( \frac{1}{a_n} ((b - S_{n-1}) \vee M_{n-1}) \right)$$

Return  $Z_1(n, b)$

**procedure** ESTIMATOR2( $n, b$ )

Generate a sample of  $J_n$  such that for  $j = 1, \dots, n-1$ ,  $P\{J_n = j\} = q(j, n) := a_j / \sum_{i=1}^{n-1} a_i$

For  $1 \leq i \leq n, i \neq J_n$  simulate  $X_i$  independently from the distribution  $F(\cdot)$

Let  $S_n^{(-J_n)} = \sum_{i=1, i \neq J_n}^n a_i X_i$ ,  $M_n^{(-J_n)} = \max\{a_i X_i : i \leq n, i \neq J_n\}$ ,

$$Z_{2,1}(n, b) = \bar{F} \left( \frac{1}{a_{J_n}} \left( (b - S_n^{(-J_n)}) \vee M_n^{(-J_n)} \right) \right),$$

$$Z_{2,2}(n, b) = \bar{F} \left( \frac{1}{a_{J_n}} \left( (b - S_{n-1}^{(-J_n)}) \vee M_n^{(-J_n)} \right) \right) \text{ and}$$

$$Z_2(n, b) = \frac{Z_{2,1}(n, b) - Z_{2,2}(n, b)}{q(J_n, n)}.$$

Return  $Z_2(n, b)$

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**Algorithm 2** Given  $b > 0$ , the aim is to efficiently simulate  $P\{S > b\}$

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Generate a sample of  $N$  such that  $P\{N = n\} = p_n$ , for  $n \geq 1$

Let  $Z_{\text{loc}}(N, b) = \text{LOCALSIMULATION}(N, b)$

Let

$$Z(b) = \frac{Z_{\text{loc}}(N, b)}{p_N}$$

Return  $Z(b)$

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*Proof.* Since  $E[Z_{\text{loc}}(n, b)] = P\{S_n > b\} - P\{S_{n-1} > b\}$  for every  $n$  and  $b$ ,

$$\begin{aligned} E[Z(b)] &= E \left[ E \left[ \frac{Z_{\text{loc}}(N, b)}{p_N} \mid N \right] \right] \\ &= E \left[ \frac{P\{S_N > b \mid N\} - P\{S_{N-1} > b \mid N\}}{p_N} \right] \\ &= \sum_n P\{N = n\} \frac{P\{S_n > b\} - P\{S_{n-1} > b\}}{p_n}. \end{aligned}$$



Since  $P\{N = n\} = p_n$ , it is immediate that,

$$E[Z(b)] = \sum_n [P\{S_n > b\} - P\{S_{n-1} > b\}] = \lim_n P\{S_n > b\}.$$

Since  $S_n \rightarrow S$  almost surely, as  $n \rightarrow \infty$ ,  $\lim_n P\{S_n > b\}$  equals  $P\{S > b\}$ . Thus, we have that the estimators  $Z(b)$  are unbiased.  $\square$

Theorem 3 above re-emphasizes the fact that  $Z(b)$  returned by Algorithm 2 is unbiased in the estimation of  $P\{S > b\}$  for every choice of  $(p_n : n \geq 1)$  satisfying  $p_n > 0$  and  $\sum_n p_n = 1$ . However, for our simulation procedure, we take

$$(9) \quad p_n := c_b \left( a_n^\alpha + \frac{a_n}{b^r} \right),$$

for some  $r \geq 1$ . As one can infer from the variance analysis in Section 4, the choice of  $(p_n : n \geq 1)$  as in (9) is the smallest choice that makes the ratio  $E[Z_{\text{loc}}^2(n, b)] / p_n^2$  uniformly bounded by a positive constant that is not dependent on  $n$ .

#### 4. ANALYSIS OF VARIANCE OF $Z(b)$

The aim of this section is to prove the following theorem when Assumptions 1 and 2 are in force:

**Theorem 4.** *For the choice of probabilities  $(p_n : n \geq 1)$  as in (9), if  $r$  is taken larger than 1, the family of estimators  $(Z(b) : b > 0)$  returned by Algorithm 2 has vanishing relative error, asymptotically, as  $b \rightarrow \infty$ . In other words,*

$$\lim_{b \rightarrow \infty} \frac{E[Z^2(b)]}{P\{S > b\}^2} = 1.$$

To prove that the estimators  $Z(b)$  have low variance asymptotically as in the statement of Theorem 4, we need to establish that  $E[Z_{\text{loc}}^2(n, b)]$  is comparable to that of  $p_n^2 \bar{F}^2(b)$ , which is challenging because proving such a proposition will have to establish that  $E[Z_{\text{loc}}^2(n, b)]$  is low with respect to two rarity parameters  $n$  and  $b$ . We accomplish this in the following section.

**4.1. Uniform bounds on variance of local estimators.** To obtain bounds on variance of estimators  $Z_{\text{loc}}(n, b)$ , we separately analyse the second moments of  $Z_1(n, b)$  and  $Z_2(n, b)$  (defined in Algorithm 1) below. Proposition 5 which is stated below and proved in the appendix will be useful in the analysis.

**Proposition 5.** *Under Assumptions 1 and 2,*

$$P\left\{S_n^{(-j)} > b, M_n^{(-j)} \leq \frac{b}{k}\right\} \leq \exp(k + o(1)) \left(\frac{\sum_i a_i^\alpha}{k} \bar{F}\left(\frac{b}{k}\right)\right)^k, \text{ as } b \rightarrow \infty$$

*uniformly in  $n$ , for every  $j \leq n$  and  $k > 1$ .*

**Remark 1.** For large values of  $b$ , Proposition 5 roughly captures the idea that when the maximum of the increments are constrained, for example, to be smaller than  $b/2$ , the likely way for a heavy-tailed sum to become larger than  $b$  is by having two large increments roughly of size  $b/2$ . Though  $k$  being an integer helps in understanding the upper bound in Proposition 5 in terms of the number of jumps, one can check from the proof of Proposition 5 that the upper bound holds true for  $k$  being any real number larger than 1.

4.1.1. *Analysis of  $Z_1(n, b)$ .* Recall that

$$Z_1(n, b) := \bar{F} \left( \frac{1}{a_n} ((b - S_{n-1}) \vee M_{n-1}) \right).$$

To upper bound second moment of  $Z_1(n, b)$ , we consider the following two quantities:

$$\begin{aligned} I_1(n, b) &:= E \left[ Z_1^2(n, b); (b - S_{n-1}) \vee M_{n-1} \geq \gamma b \right] \text{ and} \\ I_2(n, b) &:= E \left[ Z_1^2(n, b); (b - S_{n-1}) \vee M_{n-1} < \gamma b \right] \end{aligned}$$

for  $\gamma \in (0, 1)$ .

**Lemma 6.** *Under Assumptions 1 and 2,*

$$\overline{\lim}_{b \rightarrow \infty} \sup_{n > 1} \frac{I_1(n, b)}{(a_n^{\alpha-\delta} \bar{F}(b))^2} \leq (1 + \delta)^2$$

for every  $\delta > 0$  and  $\gamma \in (0, 1)$ .

*Proof.* From the definition of  $Z_1(n, b)$ , it is immediate that

$$I_1(n, b) \leq \bar{F}^2(b) E \left[ \frac{\bar{F}^2 \left( \frac{b}{a_n} \left( \left( 1 - \frac{S_{n-1}}{b} \right) \vee \gamma \right) \right)}{\bar{F}^2(b)} \right].$$

Since  $\bar{F}(x) = x^{-\alpha+o(1)}$ , given  $\delta > 0$ , for  $b$  large enough, because of (1), we have that for every  $n$ ,

$$\frac{\bar{F} \left( \frac{b}{a_n} \left( \left( 1 - \frac{S_{n-1}}{b} \right) \vee \gamma \right) \right)}{\bar{F}(b)} \leq (1 + \delta) a_n^{\alpha-\delta} h \left( \frac{S_{n-1}}{b} \right),$$

where  $h(x) = ((1 - x) \vee \gamma)^{-(\alpha+\delta)}$ . Therefore,

$$\sup_{n \geq 1} \frac{I_1(n, b)}{(a_n^{\alpha-\delta} \bar{F}(b))^2} \leq (1 + \delta)^2 \sup_{n \geq 1} E \left[ h^2 \left( \frac{S_{n-1}}{b} \right) \right].$$

Since  $h(\cdot)$  is a non-decreasing function, it is immediate that

$$\sup_{n \geq 1} \frac{I_1(n, b)}{(a_n^{\alpha-\delta} \bar{F}(b))^2} \leq (1 + \delta)^2 E \left[ h^2 \left( \frac{\sum_n a_n X_n^+}{b} \right) \right],$$

where  $x^+ := \max\{x, 0\}$  for  $x \in \mathbb{R}$ . The following observations are in order:

- 1)  $h(\cdot)$  is bounded
- 2) The random variable  $\sum_n a_n X_n^+$  is proper (this is because  $\sum_n a_n < \infty$  and hence a consequence of Kolmogorov's three-series theorem). Therefore,  $b^{-1} \sum_n a_n X_n^+ \rightarrow 0$  almost surely, as  $b \rightarrow \infty$ .

Then because of bounded convergence,

$$E \left[ h^2 \left( \frac{\sum_n a_n X_n^+}{b} \right) \right] \rightarrow 1, \text{ as } b \rightarrow \infty.$$

Thus, for every  $\delta > 0$ , we have that

$$\overline{\lim}_{b \rightarrow \infty} \sup_{n \geq 1} \frac{I_1(n, b)}{\left(a_n^{\alpha-\delta} \bar{F}(b)\right)^2} \leq (1 + \delta)^2.$$

□

**Lemma 7.** *Under Assumptions 1 and 2, there exists  $\gamma$  in  $(0, 1)$  such that*

$$\overline{\lim}_{b \rightarrow \infty} \sup_{n \geq 1} \frac{E[I_2(n, b)]}{(p_n \bar{F}(b))^2} = 0.$$

*Proof.* Observe that  $(b - S_{n-1}) \vee M_{n-1}$  is at least  $b/n$ , and this is achieved with equality when  $a_i X_i = b/n$  for every  $i < n$ . Therefore,

$$\begin{aligned} I_2(n, b) &:= E[Z_1^2(n, b); S_{n-1} > (1 - \gamma)b, M_{n-1} \leq \gamma b] \\ (10) \quad &\leq \bar{F}^2\left(\frac{b}{na_n}\right) P\{S_{n-1} > (1 - \gamma)b, M_{n-1} \leq \gamma b\} \end{aligned}$$

Since  $\sum_n na_n < \infty$ ,  $\sup_n n^2 a_n$  exists. Additionally, since  $\bar{F}(x) = x^{-\alpha} L(x) = x^{-\alpha+o(1)}$ , one can write

$$\bar{F}^2\left(\frac{b}{na_n}\right) \leq (1 + o(1)) \left(\frac{na_n}{b}\right)^{2(\alpha+o(1))} \leq (1 + o(1)) \left(\sup_n n^2 a_n\right)^{\alpha+o(1)} \left(\frac{a_n}{b^2}\right)^{\alpha+o(1)}$$

uniformly in  $n$ , as  $b \rightarrow \infty$ . Further, it follows from Proposition 5 that for every  $n$ ,

$$P\{S_{n-1} > (1 - \gamma)b, M_{n-1} \leq \gamma b\} \leq C_\gamma \bar{F}^{\frac{1-\gamma}{\gamma}}(b),$$

for some suitable constant  $C_\gamma > 0$  and all  $b$  large enough. Recall the definition of  $p_n$  in (9). Since  $p_n \geq c_b a_n b^{-r}$ , it follows from (10) that

$$\frac{I_2(n, b)}{(p_n \bar{F}(b))^2} \leq C_\gamma \left(\sup_n n^2 a_n\right)^{\alpha+o(1)} \left(\frac{a_n}{b^2}\right)^{\alpha+o(1)} \frac{b^{2r} \bar{F}^{\frac{1-\gamma}{\gamma}}(b)}{c_b^2 a_n^2 \bar{F}^2(b)},$$

uniformly in  $n$ , as  $b \rightarrow \infty$ . Since  $\alpha > 2$  and  $c_b \sim 1/\sum_n a_n^\alpha$  as  $b \rightarrow \infty$ , it follows that

$$\overline{\lim}_{b \rightarrow \infty} \sup_{n \geq 1} \frac{I_2(n, b)}{(p_n \bar{F}(b))^2} = 0$$

for any choice of  $\gamma < 1/3$ . □

Recall that  $p_n \geq c_b a_n^\alpha$ . Since  $E[Z_1^2(n, b)]$  is the sum of  $I_1(n, b)$  and  $I_2(n, b)$ ,

$$\frac{E[Z_1^2(n, b)]}{(p_n^{1-\delta} \bar{F}(b))^2} \leq \frac{I_1(n, b)}{\left((c_b a_n^\alpha)^{1-\delta} \bar{F}(b)\right)^2} + \frac{I_2(n, b)}{(p_n \bar{F}(b))^2}$$

for every  $n$  and  $b$ . Further, we have that  $c_b \sim 1/\sum_n a_n^\alpha$  as  $b \rightarrow \infty$ . Then the following is a simple consequence of Lemmas 6 and 7:

$$(11) \quad \overline{\lim}_{b \rightarrow \infty} \sup_{n \geq 1} \frac{E[Z_1^2(n, b)]}{(p_n^{1-\delta} \bar{F}(b))^2} \leq \lim_{b \rightarrow \infty} \frac{1}{c_b^{2(1-\delta)}} \times (1 + \delta)^2 + 0 = (1 + \delta)^2 \left(\sum_n a_n^\alpha\right)^{2(1-\delta)}.$$

4.1.2. *Analysis of  $Z_2(n, b)$ .* Recall that

$$Z_2(n, b) = \frac{1}{q(J_n, n)} \left[ \bar{F} \left( \frac{\xi_1}{a_{J_n}} \right) - \bar{F} \left( \frac{\xi_2}{a_{J_n}} \right) \right],$$

where

$$\xi_1 := \left( b - S_n^{(-J_n)} \right) \vee M_n^{(-J_n)} \text{ and } \xi_2 := \left( b - S_{n-1}^{(-J_n)} \right) \vee M_n^{(-J_n)}.$$

To upper bound the second moment of  $Z_2(n, b)$ , we need the following non-restrictive smoothness assumption on  $\bar{F}(\cdot)$  :

**Assumption 3.** *There exists a  $t_0$  such that the slowly varying function  $L(\cdot)$  in  $\bar{F}(x) = L(x)x^{-\alpha}$  is continuously differentiable for all  $t > t_0$ . Further,  $F(\cdot)$  is absolutely continuous, the corresponding probability density function  $f(\cdot)$  is bounded, and there exists a constant  $c > 0$  such that*

$$(12) \quad \bar{F}(x) - \bar{F}(y) \leq c(y - x) \frac{\bar{F}(x)}{x}$$

for all  $y > x \geq t_0$

One sufficient condition for (12) to hold is that the slowly varying function  $L(\cdot)$  in  $\bar{F}(x) = L(x)x^{-\alpha}$  satisfies

$$L'(t) = o \left( \frac{L(t)}{t} \right) \quad \text{as } t \rightarrow \infty.$$

Similar to the analysis of second moment  $Z_1(n, b)$ , we upper bound  $E[Z_2^2(n, b)]$  via the following two terms: Let

$$J_1(n, b) := E \left[ Z_2^2(n, b); \xi_1 \wedge \xi_2 \geq \left( a_{J_n}^\eta \wedge \gamma \right) b \right] \text{ and}$$

$$J_2(n, b) := E \left[ Z_2^2(n, b); \xi_1 \wedge \xi_2 < \left( a_{J_n}^\eta \wedge \gamma \right) b \right]$$

for some fixed  $\eta$  and  $\gamma$  in  $(0, 1)$ .

**Lemma 8.** *Under Assumptions 1, 2 and 3,*

$$\lim_{b \rightarrow \infty} \sup_n \frac{J_1(n, b)}{(p_n \bar{F}(b))^2} = 0$$

for every  $\gamma$  in  $(0, 1)$  and some  $\eta$  in  $(0, 1)$ .

*Proof.* Observe that  $|\xi_1 - \xi_2| \leq a_n |X_n|$ . Therefore, whenever both  $\xi_1/a_{J_n}$  and  $\xi_2/a_{J_n}$  are larger than  $t_0$ , due to (12),

$$Z_2^2(n, b) \leq \frac{c^2}{q^2(J_n, n)} \frac{a_n^2 X_n^2}{a_{J_n}^2} \frac{a_{J_n}^2}{(\xi_1 \wedge \xi_2)^2} \bar{F}^2 \left( \frac{\xi_1 \wedge \xi_2}{a_{J_n}} \right).$$

As a consequence, we have for every  $n$ ,

$$J_1(n, b) \leq E \left[ Z_2^2(n, b); \xi_1 \wedge \xi_2 \geq \gamma a_{J_n}^\eta b \right] \leq c^2 a_n^2 E[X_n^2] E \left[ \frac{1}{q^2(J_n, n) a_{J_n}^2} \frac{a_{J_n}^{2(1-\eta)}}{\gamma^2 b^2} \bar{F}^2 \left( \frac{\gamma b}{a_{J_n}^{1-\eta}} \right) \right].$$

Then given  $\delta > 0$ , for large values of  $b$ , due to (1),

$$\bar{F} \left( \frac{\gamma b}{a_{J_n}^{1-\eta}} \right) \leq (1 + \delta) \left( \frac{a_{J_n}^{1-\eta}}{\gamma b} \right)^{\alpha - \delta}.$$

Further, since  $q(j, n) = a_j / \sum_{i=1}^n a_i$ ,

$$J_1(n, b) \leq (1 + \delta)^2 \frac{c^2 a_n^2}{\gamma^{2(\alpha-\delta+1)}} \left( \sum_{i=1}^n a_i \right)^2 E[X_n^2] E \left[ \frac{a_{J_n}^\nu}{b^{2(\alpha-\delta+1)}} \right],$$

where  $\nu := 2(1 - \eta)(\alpha - \delta + 1) - 4$ . If we choose  $\eta < (\alpha - \delta - 1)/(\alpha - \delta + 1)$ , then  $\nu$  is positive. Additionally, since  $p_n \geq c_b a_n b^{-r}$  (for some  $r < 1$ ),

$$\sup_n \frac{J_1(n, b)}{(p_n \bar{F}(b))^2} \leq (1 + \delta)^2 \frac{c^2}{c_b^2 \gamma^{2(\alpha-\delta+1)}} \left( \sum_{i=1}^\infty a_i \right)^2 E[X^2] \frac{b^{-2(\alpha-\delta-r+1)}}{\bar{F}^2(b)}.$$

As  $\bar{F}(x) \geq (1 - \delta)x^{-\alpha-\delta}$  for large values of  $x$ , it follows that

$$\lim_{b \rightarrow \infty} \sup_n \frac{J_1(n, b)}{p_n^2 \bar{F}^2(b)} = 0$$

for any  $\delta$  smaller than  $(1 - r)/2$ , and this proves the claim.  $\square$

For the analysis of  $J_2(n, b)$ , we define

$$\kappa := \sup \left\{ k : \lim_n n^k a_n < \infty \right\}$$

and separately analyse the cases  $\kappa < \infty$  and  $\kappa = \infty$ . If  $a_n$  is, for example, polynomially decaying with respect to  $n$ , then  $\kappa$  happens to be finite. Whereas if  $a_n$  is exponentially decaying with respect to  $n$ , then  $\kappa$  is infinite. The analysis for the two cases differ, and are presented below in Lemmas 9 and 10.

**Lemma 9.** *If  $\kappa = \infty$ , then under Assumptions 1, 2 and 3,*

$$\lim_{b \rightarrow \infty} \sup_n \frac{J_2(n, b)}{\left( n^{\frac{2}{\eta}} p_n \bar{F}(b) \right)^2} = 0$$

for some  $\gamma$  in  $(0, 1)$  and every  $\eta$  in  $(0, 1)$ .

*Proof.* Due to mean value theorem,

$$Z_2(n, b) = \frac{1}{q(J_n, n)} \frac{\xi_1 - \xi_2}{a_{J_n}} f \left( \frac{\zeta}{a_{J_n}} \right)$$

for some  $\zeta$  between  $\xi_1$  and  $\xi_2$ . Here recall that  $f(\cdot)$  is the probability density corresponding to the distribution  $F(\cdot)$ . Since  $|\xi_1 - \xi_2| \leq a_n |X_n|$ , it follows from the definition of  $J_2(n, b)$  that

$$J_2(n, b) \leq E \left[ \frac{1}{q^2(J_n, n)} \frac{a_n^2 X_n^2}{a_{J_n}^2} f^2 \left( \frac{\zeta}{a_{J_n}} \right); \xi_1 \wedge \xi_2 < \left( a_{J_n}^\eta \wedge \gamma \right) \right].$$

Recall that  $q(j, n) = a_j / \sum_{i=1}^n a_i$ . Then, due to Hölder's inequality,

$$(13) \quad \frac{J_2(n, b)}{a_n^2} \leq \left( \sum_{i=1}^n a_i \right)^2 E \left[ X_n^{2p} f^{2p} \left( \frac{\zeta}{a_{J_n}} \right) \right]^{\frac{1}{p}} E \left[ \frac{1}{a_{J_n}^{4q}}; \xi_1 \wedge \xi_2 < \left( a_{J_n}^\eta \wedge \gamma \right) b \right]^{\frac{1}{q}}$$

for some  $p, q > 1$  satisfying  $p^{-1} + q^{-1} = 1$  and  $E[X^{2p}] < \infty$ . See that, as in the proof of Lemma 7,  $\xi_1 \wedge \xi_2$  is at least  $b/n$ . Therefore,

$$\begin{aligned} E \left[ \frac{1}{a_{J_n}^{4q}}; \xi_1 \wedge \xi_2 < (a_{J_n}^\eta \wedge \gamma) b \right] &= E \left[ \left( \frac{b}{\xi_1 \wedge \xi_2} \right)^{\frac{4q}{\eta}}; \xi_1 \wedge \xi_2 < (a_{J_n}^\eta \wedge \gamma) b \right] \\ &\leq n^{\frac{4q}{\eta}} P \left\{ \xi_1 \wedge \xi_2 < (a_{J_n}^\eta \wedge \gamma) b \right\}. \end{aligned}$$

From the definition of  $\xi_1$  and  $\xi_2$ , it is immediate that for every  $n$ ,

$$\begin{aligned} P \left\{ \xi_1 \wedge \xi_2 < (a_{J_n}^\eta \wedge \gamma) b \mid J_n \right\} &\leq P \left\{ S_n^{(-J_n)} \vee S_{n-1}^{(-J_n)} > (1 - \gamma)b, M_n^{(-J_n)} \leq \gamma b \mid J_n \right\} \\ (14) \quad &\leq c_\gamma \bar{F}^{\frac{1-\gamma}{\gamma}}(b) \end{aligned}$$

for some constant  $c_\gamma$  and all  $b$  large enough, because of union bound and Proposition 5. Further, recall that  $p_n \geq c_b a_n b^{-r}$ ,  $E[X^{2p}]$  is finite, and  $f(\cdot)$  is bounded. These observations, in conjunction with (13), result in

$$\sup_{n \geq 1} \frac{J_2(n, b)}{\left( n^{\frac{2}{\eta}} p_n \bar{F}(b) \right)^2} = O \left( \frac{b^{2r} \bar{F}^{\frac{1-\gamma}{\gamma q}}(b)}{\bar{F}^2(b)} \right), \text{ as } b \rightarrow \infty.$$

Given  $r < 1$  and  $q$ , one can choose  $\gamma$  suitably so that  $b^{2r} \bar{F}^{\frac{1-\gamma}{\gamma q}}(b)$  vanishes as  $b \rightarrow \infty$ . This proves the claim.  $\square$

**Lemma 10.** *If  $\kappa < \infty$ , then under Assumptions 1, 2 and 3,*

$$\lim_{b \rightarrow \infty} \sup_n \frac{J_2(n, b)}{(p_n \bar{F}(b))^2} = 0$$

*for some  $\gamma$  in  $(0, 1)$  and every  $\eta$  in  $(0, 1)$ .*

*Proof.* Observe that the argument leading to (13) in the proof of Lemma 9 holds irrespective of whether  $\kappa$  is finite or not. To proceed further, see that

$$(15) \quad E \left[ \frac{1}{a_{J_n}^{4q}}; \xi_1 \wedge \xi_2 < (a_{J_n}^\eta \wedge \gamma) b \right] = E \left[ \frac{1}{a_{J_n}^{4q}} P \left\{ \xi_1 \wedge \xi_2 < (a_{J_n}^\eta \wedge \gamma) b \mid J_n \right\} \right].$$

It follows from the definition of  $\xi_1$  and  $\xi_2$  that

$$P \left\{ \xi_1 \wedge \xi_2 < (a_{J_n}^\eta \wedge \gamma) b \mid J_n \right\} \leq P \left\{ M_n^{(-J_n)} < a_{J_n}^\eta b \mid J_n \right\} = \prod_{i=1, i \neq J_n}^n F \left( \frac{a_{J_n}^\eta b}{a_i} \right).$$

For any fixed  $k > \kappa$ , there exists a positive constant  $\tilde{c}_k$  such that  $n^k a_n \geq \tilde{c}_k$  for all  $n$ . Then

$$P \left\{ \xi_1 \wedge \xi_2 < (a_{J_n}^\eta \wedge \gamma) b \mid J_n \right\} \leq \prod_{i=1, i \neq J_n}^n F \left( \frac{i^k a_{J_n}^\eta b}{\tilde{c}_k} \right) \leq F(1) \left( \frac{\tilde{c}_k}{a_{J_n}^\eta b} \right)^{\frac{1}{k} - 2},$$

where we have simply excluded the last  $n - \lceil (\tilde{c}_k / (a_{J_n}^\eta b))^{1/k} \rceil$  terms in the product to get an upper bound. This inequality, along with (14), results in the following loose bound which is enough for our purposes:

$$P \left\{ \xi_1 \wedge \xi_2 < (a_{J_n}^\eta \wedge \gamma) b \mid J_n \right\} \leq c F(1)^{\frac{1}{2} \left( \frac{\tilde{c}_k}{a_{J_n}^\eta b} \right)^{\frac{1}{k}} - 1} \bar{F}^{\frac{1-\gamma}{2\gamma}}(b),$$

for some constant  $c > 0$ . Using this in (15), we have that

$$E \left[ \frac{1}{a_{J_n}^{4q}}; \xi_1 \wedge \xi_2 < \left( a_{J_n}^\eta \wedge \gamma \right) b \right] \leq c b^{\frac{4q}{\eta}} E \left[ \frac{1}{(a_{J_n}^\eta b)^{\frac{4q}{\eta}}} F(1)^{\frac{1}{2} \left( \frac{\bar{c}_k}{a_{J_n}^\eta b} \right)^{\frac{1}{k}} - 1} \right] \bar{F}^{\frac{1-\gamma}{2\gamma}}(b)$$

Since  $x^{\frac{4qk}{\eta}} F(1)^{x-1}$  is bounded for positive values of  $x$ , the expectation term in the right hand side of the above equation is finite. Further,  $p_n \geq c_b a_n b^{-r}$ . As a consequence, we have from (13) that

$$\sup_{n \geq 1} \frac{J_2(n, b)}{(p_n \bar{F}(b))^2} = O \left( b^{\frac{4}{\eta} + 2r} \frac{\bar{F}^{\frac{1-\gamma}{2q\gamma}}(b)}{\bar{F}^2(b)} \right),$$

which, for suitably chosen  $\gamma$ , vanishes to 0 as  $b \rightarrow \infty$ . This concludes the proof.  $\square$

Since  $E[Z_2^2(n, b)]$  is the sum of  $J_1(n, b)$  and  $J_2(n, b)$ , when  $\kappa = \infty$ , due to Lemmas 8 and 9, one can choose  $\eta$  and  $\gamma$  in  $(0, 1)$  such that

$$(16) \quad \overline{\lim}_{b \rightarrow \infty} \sup_{n \geq 1} \frac{E[Z_2^2(n, b)]}{\left( n^{\frac{2}{\eta}} p_n \bar{F}(b) \right)^2} = 0.$$

Similarly, when  $\kappa < \infty$ , due to Lemmas 8 and 10,

$$(17) \quad \overline{\lim}_{b \rightarrow \infty} \sup_{n \geq 1} \frac{E[Z_2^2(n, b)]}{(p_n \bar{F}(b))^2} = 0.$$

**4.2. Proof of Theorem 4.** Recall that

$$Z(b) = \frac{Z_{\text{loc}}(N, b)}{p_N} = \frac{Z_1(N, b) + Z_2(N, b)}{p_N}.$$

Therefore,

$$\frac{E[Z^2(b)]}{\bar{F}^2(b)} = E \left[ \frac{Z_1^2(N, b)}{p_N^2 \bar{F}^2(b)} \right] + E \left[ \frac{Z_2^2(N, b)}{p_N^2 \bar{F}^2(b)} \right] + E \left[ \frac{Z_1(N, b)}{p_N \bar{F}(b)} \right] E \left[ \frac{Z_2(N, b)}{p_N \bar{F}(b)} \right].$$

Then due to Jensen's inequality,

$$(18) \quad \frac{E[Z^2(b)]}{\bar{F}^2(b)} \leq E \left[ \frac{Z_1^2(N, b)}{p_N^2 \bar{F}^2(b)} \right] + E \left[ \frac{Z_2^2(N, b)}{p_N^2 \bar{F}^2(b)} \right] + \sqrt{E \left[ \frac{Z_1^2(N, b)}{p_N^2 \bar{F}^2(b)} \right]} \sqrt{E \left[ \frac{Z_2^2(N, b)}{p_N^2 \bar{F}^2(b)} \right]}.$$

Now consider, for example, the first term in the right hand side of the above inequality. Due to the uniform convergence result on  $E[Z_1^2(n, b)]$  in (11), there exists a constant  $c_1$  such that

$$\frac{E[Z_1^2(N, b) \mid N]}{(p_N \bar{F}(b))^2} \leq c_1 (1 + \delta)^2 p_N^{-2\delta} \left( \sum_n a_n^\alpha \right)^{2(1-\delta)}$$

for every  $\delta$  and  $b$ . Since  $\sum_n n a_n$  exists,  $E p_N^{-2\delta} < \infty$  for all  $\delta$  small enough. As  $\delta$  can be arbitrarily small, due to reverse Fatou's lemma, it follows from (11) that

$$(19) \quad \overline{\lim}_{b \rightarrow \infty} E \left[ \frac{Z_1^2(N, b)}{p_N^2 \bar{F}^2(b)} \right] \leq E \left[ \overline{\lim}_{b \rightarrow \infty} \frac{E[Z_1^2(N, b) \mid N]}{(p_N \bar{F}(b))^2} \right] \leq \left( \sum_n a_n^\alpha \right)^2.$$

Similarly, one can conclude from (16) and (17) that for every  $b$ ,

$$\frac{E[Z_2^2(N, b) | N]}{(p_N \bar{F}(b))^2} \leq \begin{cases} c_2 N^{\frac{4}{\eta}} & \text{if } \kappa = \infty \\ c_2 & \text{if } \kappa < \infty. \end{cases}$$

for some constant  $c_2$ . Observe that  $EN^{\frac{4}{\eta}} < \infty$  for any fixed  $\eta$  because when  $\kappa = \infty$ ,  $p_n$  is exponentially decaying with respect to  $n$ . Then as a consequence of (16) and (17), due to dominated convergence,

$$\lim_{b \rightarrow \infty} E \left[ \frac{Z_2^2(N, b)}{p_N^2 \bar{F}^2(b)} \right] = E \left[ \lim_{b \rightarrow \infty} \frac{E[Z_2^2(N, b) | N]}{(p_N \bar{F}(b))^2} \right] = 0.$$

This conclusion, along with (18) and (19), results in

$$\lim_{b \rightarrow \infty} \frac{E[Z^2(b)]}{\bar{F}^2(b)} \leq \left( \sum_n a_n^\alpha \right)^2.$$

Further,  $P\{S > b\} \sim \sum_n a_n^\alpha \bar{F}(b)$  as  $b \rightarrow \infty$ . Therefore,

$$\lim_{b \rightarrow \infty} \frac{E[Z^2(b)]}{P\{S > b\}^2} \leq 1.$$

Additionally, since  $Z(b)$  is an unbiased estimator of  $P\{S > b\}$ ,  $E[Z^2(b)]$  must be larger than  $P\{S > b\}^2$  because of Jensen's inequality. This proves the theorem.  $\square$

**4.3. A note on computational complexity of the simulation procedure.** Given  $b > 0$ , our objective has been to devise an algorithm that returns a number in the interval  $((1-\epsilon)P\{S > b\}, (1+\epsilon)P\{S > b\})$  with probability at least  $1 - \delta$ . In Section 3, we proposed to take average of values returned by several runs of Algorithm 2 as the estimate of  $P\{S > b\}$ . Assuming that tasks like performing basic arithmetic operations, generating uniform random numbers, evaluating  $F(x)$  at specified  $x$ , all require unit computational effort, it is immediate that each call to the procedure `LOCALSIMULATION`( $n, b$ ) expends at most  $Cn$  computational effort, for some positive constant  $C$ , irrespective of the value of  $b$ . Given  $b > 0$ , if one makes  $N_b$  calls to Algorithm 2 and returns the average of returned values of  $Z(b)$  as the overall estimate, then

- 1) the estimate lies within the desired interval with probability at least  $\epsilon^{-2} \text{CV}^2[Z(b)]/N_b$ , where  $\text{CV}[Z_b] = \text{Var}[Z_b]/E[Z_b]^2$  is the coefficient of variation of  $Z_b$ , and
- 2) the overall computational effort is at most  $CNN_b$ , where  $N$  is the auxiliary random variable drawn according to the probability mass function  $(p_n : n \geq 1)$  in Algorithm 2.

Due to Theorem 4, we have that  $\text{CV}[Z(b)] = o(1)$ , as  $b \rightarrow \infty$ . Therefore, it is enough to choose  $N_b = c\epsilon^{-2}\delta^{-1}$  for some positive constant  $c$ . Further, note that

$$E[N] = \sum_n np_n = c_b \sum_n n \left( a_n^\alpha + \frac{a_n}{b^r} \right).$$

First, observe that  $\sum_n a_n < \infty$  because of Assumption 2. Additionally, since  $c_b \sim \sum_n a_n^\alpha$  as  $b \rightarrow \infty$ , we have  $EN = O(1)$  as  $b \rightarrow \infty$ . Therefore, the overall computational effort is just  $O(1)$  as  $b \rightarrow \infty$ . Thus, despite the difficulties that the definition of  $S$  involves infinitely many random variables and  $P\{S > b\}$  is arbitrarily small for large values of  $b$ , our work establishes that one can compute  $P\{S > b\}$  without any bias by expending only a computational effort that is uniformly bounded in  $b$ .



## 5. A NUMERICAL EXAMPLE

In this section, we present the results of a numerical simulation experiment that demonstrates the efficiency of our estimator. Take  $(X_n : n \geq 1)$  to be iid copies of a Pareto random variable  $X$  satisfying  $P\{X > x\} = 1 \wedge x^{-4}$ . Additionally, take  $a_n = 0.9^n$  and let  $S = \sum_n a_n X_n$ . We use  $N = 10,000$  simulation runs to estimate  $P\{S > b\}$  for various values of  $b$  listed in Table 1. The parameter  $r$  in the choice of probabilities  $p_n$  in the expression 9 is taken to be 1. The values listed in Column 3 correspond to the estimate obtained from 10,000 runs of our simulation algorithm. It is instructive to compare the simulation estimates in Column 3 with the crude asymptotic  $\bar{F}(b) \sum_n a_n^\alpha$  listed in Column 2. The empirically observed coefficient of variation of our simulation estimators is listed in Column 5. Although it is required in the proof of Theorem 4 that  $r > 1$ , it can be inferred from Column 5 that the choice  $r = 1$  yields estimators that have coefficient of variation that decreases to 0 as  $b$  is increased.

TABLE 1. Numerical result for the simulation of  $P\{S > b\}$ - here CV denotes the empirically observed coefficient of variation based on 10,000 simulation runs

b	Asymptotic $\bar{F}(b) \sum_n a_n^\alpha$	Estimate for $P\{S > b\}$	Standard Error	CV
200	$1.19 \times 10^{-9}$	$1.49 \times 10^{-9}$	$1.61 \times 10^{-11}$	1.08
500	$3.05 \times 10^{-11}$	$3.32 \times 10^{-11}$	$1.54 \times 10^{-13}$	0.47
1000	$1.91 \times 10^{-12}$	$1.97 \times 10^{-12}$	$8.43 \times 10^{-15}$	0.42

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## APPENDIX

We present proof of Proposition 5 here in the appendix. To accomplish this we need Lemma 11 first, which is stated and proved below.

**Lemma 11.** *For any pair of sequences  $\{x_n\}, \{\phi_n\}$  satisfying  $x_n \rightarrow \infty$  and  $\phi_n x_n \rightarrow \infty$ , the integral,*

$$\int_{-\infty}^{x_n} e^{\phi_n x} F(dx) \leq 1 + c\phi_n^\kappa + e^{2\alpha} \bar{F}\left(\frac{2\alpha}{\phi_n}\right) + e^{\phi_n x_n} \bar{F}(x_n)(1 + o(1)),$$

as  $n \rightarrow \infty$ , for any  $1 < \kappa < \alpha \wedge 2$ , and some constant  $c$  which does not depend on  $n$  and  $b$ .

*Proof.* We split the region of integration into  $(-\infty, \gamma/\phi_n]$  and  $(\gamma/\phi_n, x_n]$  for some constant  $\gamma > 0$ ; the partition is such that the integrand stays bounded in the former region.

Let  $I_1 := \int_{-\infty}^{\gamma/\phi_n} e^{\phi_n x} F(dx)$  and  $I_2 := \int_{\gamma/\phi_n}^{x_n} e^{\phi_n x} F(dx)$ .

For any  $\kappa \in (1, 2]$  and  $y > 0$ , it is easily verified that

$$e^x \leq 1 + x + |x|^\kappa e^y, \quad x \in (-\infty, y].$$

Therefore,

$$\begin{aligned} I_1 &\leq \int_{-\infty}^{\gamma/\phi_n} (1 + \phi_n x + \phi_n^\kappa |x|^\kappa \exp(\phi_n \cdot \gamma/\phi_n)) F(dx) \\ &\leq \int_{-\infty}^{\gamma/\phi_n} F(dx) + \phi_n \int_{-\infty}^{\gamma/\phi_n} x F(dx) + \phi_n^\kappa e^\gamma \int_{-\infty}^{\gamma/\phi_n} |x|^\kappa F(dx) \\ &\leq \int_{-\infty}^{\infty} F(dx) + \phi_n \int_{-\infty}^{\infty} x F(dx) + \phi_n^\kappa e^\gamma \int_{-\infty}^{\infty} |x|^\kappa F(dx) \\ (20) \quad &= 1 + c\phi_n^\kappa, \end{aligned}$$

where  $c := e^\gamma \int_{-\infty}^{\infty} |x|^\kappa F(dx) < \infty$  because  $E|X|^\kappa < \infty$ ; this follows because  $\kappa < \alpha$ . We have also used  $EX = 0$  to arrive at (20). Integrating by parts for the second integral  $I_2$ :

$$\begin{aligned} I_2 &= - \int_{\gamma/\phi_n}^{x_n} e^{\phi_n x} \bar{F}(dx) \\ &= e^{\phi_n \gamma/\phi_n} \bar{F}\left(\frac{\gamma}{\phi_n}\right) - e^{\phi_n x_n} \bar{F}(x_n) + \phi_n \int_{\gamma/\phi_n}^{x_n} e^{\phi_n x} \bar{F}(x) dx \\ (21) \quad &\leq e^\gamma \bar{F}\left(\frac{\gamma}{\phi_n}\right) + I'_2, \end{aligned}$$

where,  $I'_2 := \phi_n \int_{\gamma/\phi_n}^{x_n} e^{\phi_n x} \bar{F}(x) dx$ . Now the change of variable  $u = \phi_n(x_n - x)$  results in:

$$\begin{aligned} I'_2 &= e^{\phi_n x_n} \int_0^{\phi_n x_n - \gamma} e^{-u} \bar{F}\left(x_n - \frac{u}{\phi_n}\right) du \\ (22) \quad &= e^{\phi_n x_n} \bar{F}(x_n) \int_0^{\phi_n x_n - \gamma} e^{-u} g_n(u) du, \end{aligned}$$

where,

$$g_n(u) := \frac{\bar{F}\left(x_n - \frac{u}{\phi_n}\right)}{\bar{F}(x_n)} = \frac{\bar{F}\left(x_n \left(1 - \frac{u}{\phi_n x_n}\right)\right)}{\bar{F}(x_n)}.$$

Since  $L(\cdot)$  is slowly varying and  $\phi_n x_n \rightarrow \infty$ , given any  $\delta > 0$ , it follows from (1) that,

$$(1 - \delta) \left(1 - \frac{u}{\phi_n x_n}\right)^{-\alpha + \delta} \leq g_n(u) \leq (1 + \delta) \left(1 - \frac{u}{\phi_n x_n}\right)^{-\alpha - \delta}.$$

for all  $n$  large enough. So for any fixed  $u$ , we have  $g_n(u) \rightarrow 1$  as  $n \rightarrow \infty$ . Now fix  $\delta = \frac{\alpha}{2}$ . Then for  $n$  large enough,

$$(23) \quad g_n(u) \leq \left(1 + \frac{\alpha}{2}\right) \left(1 - \frac{u}{\phi_n x_n}\right)^{-\frac{3\alpha}{2}}.$$

Let  $h(u) = (1 - u/\phi_n x_n)^{-\frac{3\alpha}{2}}$ . Since  $\log h(0) = 0$  and  $\frac{d}{du}(\log(h(u))) \leq \frac{3\alpha}{2\gamma}$  for  $0 \leq u \leq \phi_n x_n - \gamma$ , we have  $h(u) \leq \exp(3\alpha u/2\gamma)$  on the same interval. Therefore if we choose  $\gamma = 2\alpha$ , the integrand in  $I'_2$  is bounded for large enough  $n$  by an integrable function as below:

$$\begin{aligned} |e^{-u} g_n(u) \mathbf{1}(0 \leq u \leq \phi_n x_n - \gamma)| &\leq \left| e^{-u} \left(1 + \frac{\alpha}{2}\right) h(u) \mathbf{1}(0 \leq u \leq \phi_n x_n - \gamma) \right| \\ &\leq \left(1 + \frac{\alpha}{2}\right) e^{-u + \frac{3\alpha u}{2\gamma}} = \left(1 + \frac{\alpha}{2}\right) e^{-\frac{u}{4}}. \end{aligned}$$

Applying dominated convergence theorem, we get

$$\int_0^{\phi_n x_n - \gamma} e^{-u} g_n(u) du \sim 1 \text{ as } n \rightarrow \infty.$$

Since  $\int_{-\infty}^{x_n} e^{\phi_n x} F(dx) = I_1 + I_2$ , combining this result with (20), (21) and (22), completes the proof.  $\square$

*Proof of Proposition 5.* Observe that for any  $n$  and  $j$ ,

$$\left\{ M_n^{(-j)} \leq \frac{b}{k} \right\} = \bigcap_{i=1, i \neq j}^n \left\{ X_i \leq \frac{b}{ka_i} \right\}.$$

Then for any  $\theta > 0$ ,

$$P \left\{ S_n^{(-j)} > b, M_n^{(-j)} \leq \frac{b}{k} \right\} \leq \exp(-\theta b) \prod_{i=1, i \neq j}^n E \left[ \exp(\theta a_i X_i); X_i \leq \frac{b}{ka_i} \right]$$

because of a simple application of Markov's inequality. If  $\theta$  is chosen such that  $\theta b \rightarrow \infty$  as  $b \rightarrow \infty$ , from Lemma 11, we have

$$E \left[ \exp(\theta a_i X_i); X_i \leq \frac{b}{ka_i} \right] \leq 1 + c\theta^2 a_i^2 + e^{2\alpha} \bar{F} \left( \frac{2\alpha}{\theta a_i} \right) + \exp \left( \theta \frac{b}{k} \right) \bar{F} \left( \frac{b}{ka_i} \right) (1 + o(1)),$$

uniformly in  $i$ , as  $b \rightarrow \infty$ . Since  $1 + x \leq \exp(x)$ ,

$$\begin{aligned} &P \left\{ S_n^{(-j)} > b, M_n^{(-j)} \leq \frac{b}{k} \right\} \\ &\leq \exp(-\theta b) \prod_{i=1, i \neq j}^n \exp \left( c\theta^2 a_i^2 + e^{2\alpha} \bar{F} \left( \frac{2\alpha}{\theta a_i} \right) + \exp \left( \theta \frac{b}{k} \right) \bar{F} \left( \frac{b}{ka_i} \right) (1 + o(1)) \right) \\ (24) \quad &\leq \exp \left( -\theta b + c\theta^2 \sum_i a_i^2 + e^{2\alpha} \sum_i \bar{F} \left( \frac{2\alpha}{\theta a_i} \right) + \bar{F} \left( \frac{b}{k} \right) \exp \left( \theta \frac{b}{k} \right) \sum_i a_i^{\alpha-\epsilon} (1 + o(1)) \right), \end{aligned}$$

for any given  $\epsilon > 0$ , due to (1), uniformly in  $j$  and  $n$ , as  $b \rightarrow \infty$ . Observe that

$$\theta_b := -\frac{k}{b} \log \left( \frac{\sum_i a_i^\alpha}{k} \bar{F} \left( \frac{b}{k} \right) \right)$$

is the minimizer of  $-\theta b + \sum_i a_i^\alpha \bar{F}(b/k) \exp(\theta b/k)$ , and it approximately minimizes the right hand side of (24). Since  $\theta_b \searrow 0$  and  $\sum_i a_i^{\alpha-\epsilon} < \infty$  for small enough  $\epsilon$ , it follows from (1) that

$$\begin{aligned} \sum_i \bar{F} \left( \frac{2\alpha}{\theta_b a_i} \right) &\leq (1 + \epsilon) \sum_i \left( \frac{a_i}{2\alpha} \right)^{\alpha-\epsilon} \bar{F} \left( \frac{1}{\theta_b} \right) = o(\theta_b), \\ \theta_b^2 &= o(\theta_b), \text{ and } \bar{F} \left( \frac{b}{k} \right) \exp \left( \theta_b \frac{b}{k} \right) \sum_i a_i^\alpha = k, \end{aligned}$$

as  $b \rightarrow \infty$ . Therefore, uniformly for every  $n$  and  $j \leq n$ ,

$$\begin{aligned} P \left\{ S_n^{(-j)} > b, M_n^{(-j)} \leq \frac{b}{k} \right\} &\leq \exp \left( k \log \left( \frac{\sum_i a_i^\alpha}{k} \bar{F} \left( \frac{b}{k} \right) \right) + o(1) + k(1 + o(1)) \right) \\ &= \exp(k + o(1)) \left( \frac{\sum_i a_i^\alpha}{k} \bar{F} \left( \frac{b}{k} \right) \right)^k, \end{aligned}$$

as  $b \rightarrow \infty$ . This proves the claim. □

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